Reduced order model from a new stochastic decomposition of fluid flow

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Plan

Stochastic fluid dynamic model
- Classical Large Eddy approach
- Stochastic Navier-Stokes model

Reduced model

Results

Conclusion

CLASSICAL LARGE EDDY APPROACH

Numericians approach :

$$\frac{\partial \overline{T}}{\partial t} + \nabla \cdot \left(\overline{T} \bar{u} \right) = \nabla \cdot \left(-\overline{u' T'} \right)$$

Classical Large Eddy approach

Numericians approach :

$$\frac{\partial \overline{T}}{\partial t} + \nabla \cdot \left(\overline{T}\bar{u}\right) = \nabla \cdot \left(-\overline{u'T'}\right) = \frac{\sigma^2}{2} \triangle \overline{T}$$

Classical Large Eddy approach

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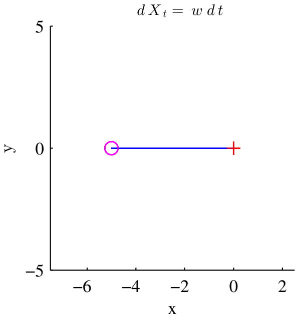

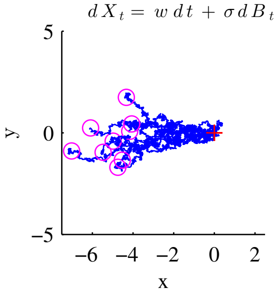

$\frac{dB_t}{dt}$ = white noise, and B_t = Brownian motion

STOCHASTIC NAVIER-STOKES MODEL

Notations :

- $u = w + \frac{\sigma dB}{dt}$
- w continuous and differentiable w.r.t. time : large scales
- $\frac{\sigma dB}{dt}$ Gaussian and discontinuous w.r.t. time : small scales

$$\sigma(x,t)\frac{dB_t}{dt} \triangleq \int_\Omega \underbrace{\breve{\sigma}(x,z,t)}_{\text{kernel}} \underbrace{\frac{dB_t}{dt}(z)}_{\text{white noise}} \, dz$$

- Local covariance matrix
$a(x,t) \triangleq \sigma(x,t)\sigma^t(x,t) = \int_\Omega \breve{\sigma}(x,z,t)\breve{\sigma}^t(x,z,t)dz$

STOCHASTIC NAVIER-STOKES MODEL

Theorem

$$\begin{aligned}\frac{\partial w_i}{\partial t} + (w^* \cdot \nabla) w_i &= \nabla \cdot \left(\frac{a}{2} \nabla w_i \right) - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \triangle w, \\ with\ \ w^* &= w - \frac{(\nabla \cdot a)^T}{2}\end{aligned}$$

Applications :

- Large eddies simulation
- Uncertainty quantification
- Filtering
- Downscaling
- Mixing diagnostics

...

Plan

Stochastic fluid dynamic model

Reduced model

- Principle / Notations
- Additional reduction : time sub-sampling
- Our approach
- Estimation of a

Results

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Principle / Notations

- Galerkin projection from POD modes :

$$u(x,t) \approx \sum_{i=0}^{n} b_i(t)\phi_i(x)$$

- Chronos : b_i
- $\dfrac{\partial u}{\partial t} = I + L(u) + C(u,u)$ (a PDE) becomes :

$$\forall i, \frac{db_i}{dt} = \left(\int_\Omega \phi_i \cdot I\right) + \sum_{p=0}^{n}\left(\int_\Omega \phi_i \cdot L(\phi_p)\right) b_p + \sum_{p,q=0}^{n}\left(\int_\Omega \phi_i \cdot C(\phi_p,\phi_q)\right) b_p b_q \qquad \text{(ODEs)}$$

ADDITIONAL REDUCTION : TIME SUB-SAMPLING

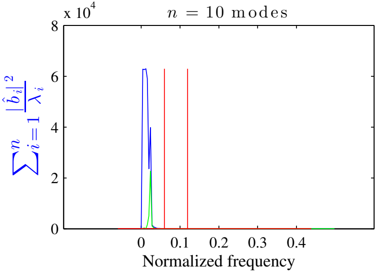
Large time scale approach

we keep only necessary information for the resolved modes :

$$\frac{db_i}{dt} = i_i + l_{\bullet i} b + b^T c_{\bullet\bullet i} b$$

Nyquist-Shanon criterion for sub-sampling :

$\frac{f_s}{2} = 2 \max_i f_{max}(b_i)$ meets Shanon.



OUR APPROACH

Our approach :

- $u = w + \frac{\sigma dB}{dt}$
- $w = \sum_{i=0}^{n} b_i \phi_i$
  (resolved modes)
- $v_{res} = u - w = \sum_{i=n+1}^{N} b_i \phi_i$ a realization of $\frac{\sigma dB}{dt}$
  (aliased $\Rightarrow$ partially decorrelated in time)

ESTIMATION OF a CONSTANT IN TIME

Local covariance matrix estimated by :

$$a(x) \approx \Delta t \frac{1}{N} \sum_{k=1}^{N} v_{res}(x, t_k) v_{res}^T(x, t_k)$$

ESTIMATION OF a DEPENDING OF TIME

We look for a decomposition of $a = \sum_{i=0}^{n} b_i(t) z_i(x)$.

Estimation based on Stochastic Calculus (Genon-Catalot 1992) :

$$z_i(x) = \int_0^T \frac{b_i}{T\lambda_i} a dt \approx \Delta t \frac{1}{N} \sum_{k=1}^{N} \frac{b_i(t_k)}{\lambda_i} v_{res}(x, t_k) v_{res}^T(x, t_k)$$

PLAN

STOCHASTIC FLUID DYNAMIC MODEL

REDUCED MODEL

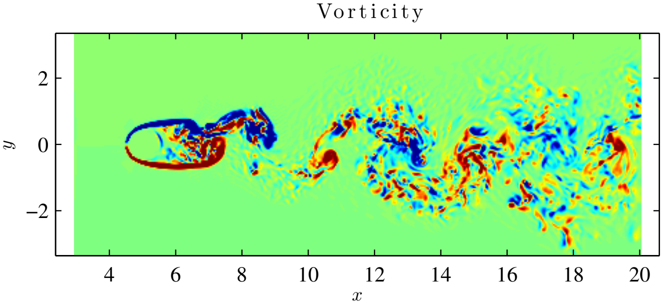
RESULTS
- Data
- Reconstruction of temporal modes

CONCLUSION

DATA

Wake behind a cylinder

- at Reynolds 3900 : 3D flow



RECONSTRUCTIONS OF TEMPORAL MODES

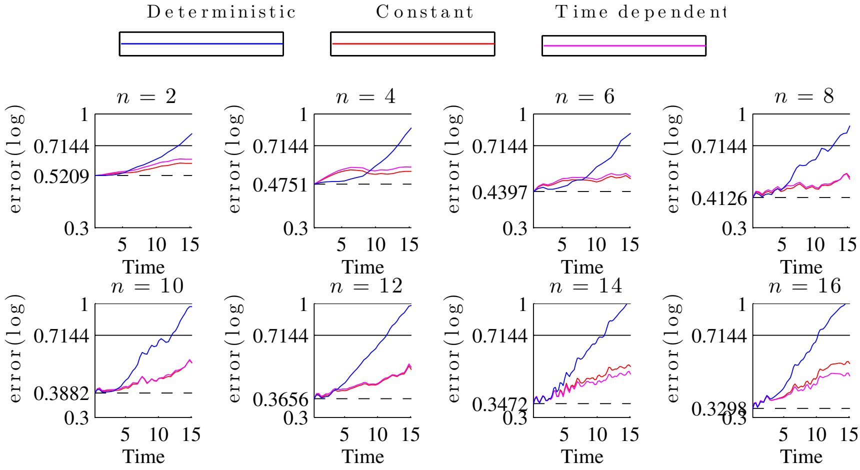


FIGURE : Re=3900 (3D flow), *a* constant and in the span of *Chronos* b_i.

PLAN

STOCHASTIC FLUID DYNAMIC MODEL

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Summary

- Complete fluid dynamics model
  based on Physics and Stochastic Calculus :
  Generalization of empirical approaches
- Local covariance matrix of the small-scales $a(x, t)$
  constant or projected on *Chronos*
- Good results without tuning :
  Stabilization and correction of the frequency drift

FURTHER WORK

- Test on DNS data with more available snapshots
- $a = \sum_{i \leqslant j} b_i b_j z_{ij}(x) \rightarrow$ Inverse problem estimation
- Generalization :
  w continuous but not differentiable w.r.t. time
  Energy conserving model for UQ

$$\frac{\partial w_i}{\partial t} + (w^* \cdot \nabla) w_i = \nabla \cdot \left(\frac{a}{2} \nabla w_i \right) - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \triangle w \underbrace{- \left(\frac{\sigma dB_t}{dt} \cdot \nabla \right) w_i}_{\text{multiplicative noise}} + \underbrace{\nu \triangle \frac{\sigma_i dB_t}{dt}}_{\text{additive noise}}$$

QUESTIONS

Thank you for your attention

ESTIMATION OF a DEPENDING OF TIME

$$\forall x \in \mathbb{R}^d,\ d\tilde{X}_t^x \triangleq \sigma(x,t) dB_t$$

$$\begin{aligned} z_i(x) &= \int_0^T \frac{b_i(t)}{\lambda_i T} a(x,t) dt \\ &= \int_0^T \frac{b_i(t)}{\lambda_i T} d < \tilde{X}^x, \left(\tilde{X}^x\right)^t >_t \quad (\text{since } a \triangleq \sigma\sigma^t) \\ &= \mathbb{P} - \lim_{\Delta t \to 0} \sum_{t_k=0}^{T} \frac{b_i(t_k)}{\lambda_i T} \left(\tilde{X}^x_{t_{k+1}} - \tilde{X}^x_{t_k}\right) \left(\tilde{X}^x_{t_{k+1}} - \tilde{X}^x_{t_k}\right)^t \end{aligned}$$

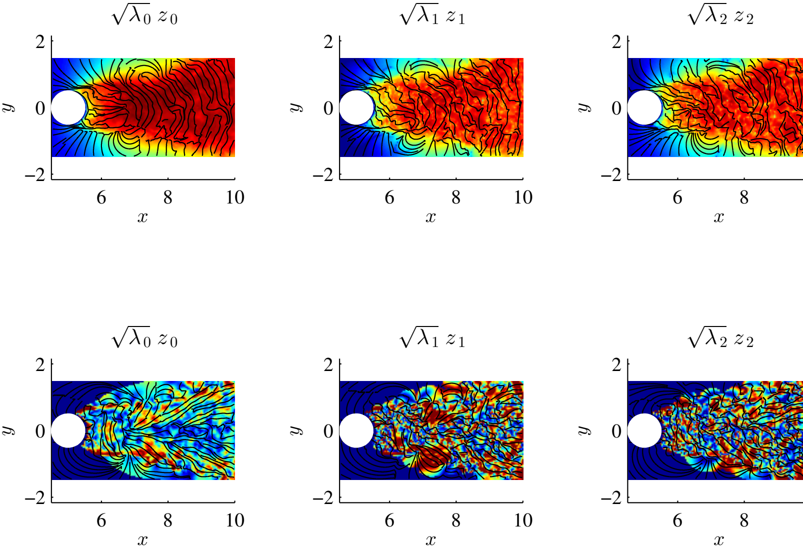
$v_{res}(x,t)\Delta t$ a realization of $d\tilde{X}_t^x \triangleq \sigma(x,t) dB_t$

- Estimator with good statistical properties (Genon-Catalot 1992) (when the projection subset is well chosen)

TURBULENCE MODES z_i : EIGENVALUES/EIGENVECTORS : RE=3900

1^{st} vector & Energy

2^{nd} vector & Anisotropy



Reconstructions of temporal modes

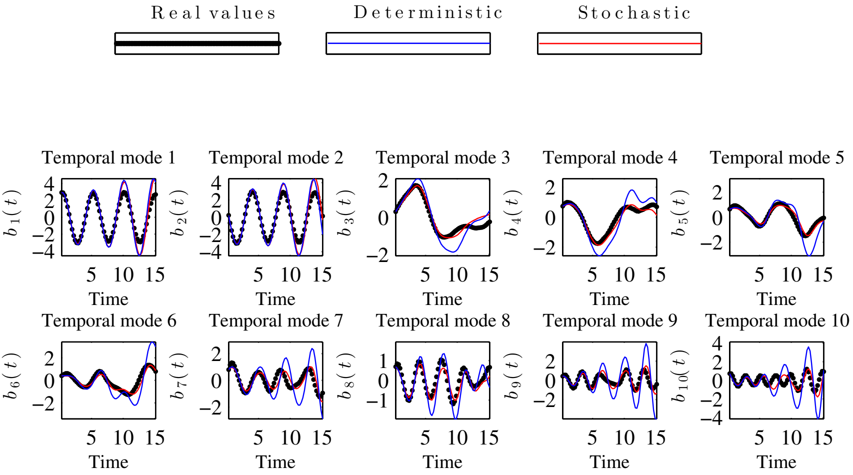


Figure : n=10, Re=3900 (3D flow), *a* constant in time.